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Numerical Investigation of One-Dimensional Visco-Plastic Model

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Third Quarterly Report
May 3, 1961-September 3, 1961

General Electric Company
Space Sciences Laboratory
King of Prussia, Pennsylvania

for DEPUTY FOR AEROSPACE

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October 1961

NUMERICAL INVESTIGATION OF ONE-DIMENSIONAL
VISCO-PLASTIC MODEL

Third Quarterly Report
May 3, 1961 - September 3, 1961

by
T. D. Riney

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October 1961

GENERAL ELECTRIC COMPANY
Space Sciences Laboratory
King of Prussia, Pennsylvania

Deputy for Aerospace
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Air Force Systems Command
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FOREWORD

This report was prepared under Air Force Contract No. AF 08(635)-1713, (U) "Theory of High Speed Impact). The work was administered under the direction of APGC (PGTWR). The project engineer was Dr. T. D. Riney.

The assistance of Mr. P. R. Chernoff of the General Electric Co. in the preparation of this report is gratefully acknowledged. He contributed to the numerical analysis and, also, programed the difference equations for machine calculations.

ABSTRACT

An explicit finite difference formulation of the equations governing the one-dimensional visco-plastic model was presented in the Second Quarterly Report. The scheme has been programmed on the IBM 7090 and exploratory calculations made for several values of the parameters μ_0 (viscosity factor), τ_0 (yield stress) and v_0 (impact velocity). Excellent results are obtained for certain ranges of the parameters; these results are discussed in detail. For larger values of μ_0 and v_0 , however, the restriction on the size of Δt is very severe. A stability analysis is performed which explains this requirement.

To circumvent this difficulty the governing equations are recast in an implicit finite difference scheme which is only valid for $\mu_0 > 0$, but which is then unconditionally stable. The method is described in detail and a flow chart is given for the machine program which is currently being written.

*Catalog cards with abstracts may be
found at the back of this publication.*

LIST OF SYMBOLS

τ_0	dynamic yield value of shear stress
μ_0	material constant, μ_0^{-1} called mobility coefficient
v_0	impact velocity
ρ_0	mass density of undisturbed visco-plastic medium
ρ	density of visco-plastic medium
B	Bingham-Oldroyd number
R	Reynold's number
t	time
x	particle label in Lagrangian coordinates
z	Eulerian coordinate in one dimensional flow
q	velocity of flow in one dimensional Eulerian formulation
$\Delta[]$	delta denotes increment of quantity
p	thermodynamic pressure
U	specific internal energy of medium
$V = (1/\rho)$	specific volume
$g(V, U)$	equation of state
μ	strain-rate dependent viscosity coefficient
τ^2	von Mises flow statistic
$[]_j^n$	value of quantity at time station n, space station j
ℓ	constant with dimensions of length
a	constant determining magnitude of artificial viscosity
Q	artificial viscosity term
S	real viscosity term

c	speed of sound in visco-plastic medium
c_0	speed of sound in undisturbed medium
$\tilde{[]}_j^n$	k-th approximation of quantity at time n, position j.
K	number of iterations performed
J	position station chosen to be in front of disturbance
$[]$	first variation of quantity
A_k, \dots, F_k	coefficients in Fourier representations
β	quantity defined in (A-16)
Λ	quantity defined in (A-16)
ϵ	quantity defined in (A-16)
γ	quantity defined in (A-16)
δ	quantity defined in (A-16)
ζ	quantity defined in (A-16)
G	amplification matrix
λ	eigenvalues of amplification matrix

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- 4 Pulse velocities at various times compared with asymptotic values
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INTRODUCTION

In the First Quarterly Report (Ref. 1) a mathematical model was proposed for the study of hypervelocity impact which takes into account the effects of material viscosity and dynamic yield strength. In this formulation the material is considered rigid if stressed below its flow point, whereas the material acts like a Newtonian viscous liquid when stressed above this value.

In the Second Quarterly Report (Ref. 2) dimensionless parameters were found which control the relative importance of the inertial, viscous and strength terms in the governing partial differential equations. Both the viscosity factor μ_0 and the dynamic yield stress τ_0 appear in these parameters. Since definitive data is not available for either of these quantities a proposal was made that exploratory calculations be performed using a one-dimensional model in which the values of μ_0 and τ_0 are varied over several orders of magnitude. For this purpose the governing equations were written in one-dimensional Lagrangian form and then recast in finite difference form in preparation for the exploratory calculations. In this scheme the dependent variables at time $(n + 1)\Delta t$ are expressed explicitly in terms of their values at time $n\Delta t$.

The present report opens with a description of some of the results of the calculations made on the IBM 7090. Very good results are obtained for certain ranges of the parameters μ_0 , τ_0 and impact velocity v_0 . For larger values of μ_0 and v_0 , however, the calculations do not converge but oscillate with increasing amplitude as n increases. A stability analysis is given which shows that this effect results from the inclusion of the viscosity which causes the usual restriction on the ratio $\Delta t/\Delta x$ to be replaced by a restriction on the ratio on $\Delta t/(\Delta x)^2$. This conclusion is verified by the preliminary calculations. To circumvent this difficulty the governing equations are written in one-dimensional Eulerian form and then recast in a finite scheme which is only valid when $\mu_0 > 0$, but which is then unconditionally stable. Here the dependent variables enter implicitly so that an iterative procedure is necessary in proceeding from time n to time $n + 1$. The method is described in detail and a flow chart is given for the calculations.

GENERAL REMARKS

(a) Choice of Parameters

In choosing trial values for μ_0 our starting point has been the values quoted by Perzyna (Ref. 3). On the basis of the results of internal energy measurements by Kolsky (Ref. 4) he computes the approximate values¹

¹ The gram-centimeter-microsecond system is used throughout this report.

Steel: $\mu_0 = 0.8 \text{ gm cm}^{-1} \text{ microsec}^{-1}$.

Copper: $\mu_0 = 0.4 \text{ gm cm}^{-1} \text{ microsec}^{-1}$.

Our calculations have all been for iron and μ_0 is assumed to be within a factor of ten of 0.8.

For mild steel Reiner (Ref. 5) quotes the static yield stress to be approximately

$$\tau_0 = 10^{-2} \text{ gm cm}^{-1} \text{ microsec.}^{-2},$$

i.e. ten kilobars. For our calculations it is the dynamic yield stress that is pertinent. The above value is assumed to be the lowest value likely, and τ_0 is varied up to one hundred times as large, i.e. one megabar.

The various combinations of assumptions for μ_0 , τ_0 and impact velocity, v_0 , are displayed in Table I. The choices of v_0 ($= 0.5, 4, 7.5 \text{ cm/microsec}$) represent the extremes and mean of the meteoroid velocity range. The values of the dimensionless parameters B_0 and R_0 are also listed for each combination of μ_0 , τ_0 , v_0 . They are seen to vary widely with the choice of combinations.

(b) Description of the Flow

If the two impacting bodies are perfect fluids, shocks are produced which propagate at a constant velocity and the jump discontinuities across the shocks remain fixed. With viscosity present energy is being dissipated and it might be expected that the disturbance would be attenuated as it propagates. In the case of one-dimensional impact between semi-infinite bodies, however, the dissipation effect is counteracted. Here there is an infinite reservoir of energy, a steady state obtains in which the rate of increase of internal energy behind the disturbance just equals the kinetic energy of the medium entering the disturbance. Consequently, the shape, magnitude and velocity of the disturbance (or "pressure pulse") does not change as it propagates.* These characteristics of the steady-state disturbance, however, are affected by the values of the parameters μ_0 , τ_0 and v_0 .

The viscosity coefficient and von Mises flow rule are (Ref. 2, p. 12)

$$\mu = \mu_0 + \sqrt{\frac{3}{4}} \frac{\rho_0}{\rho} \frac{\tau_0}{\left| \frac{\partial q}{\partial x} \right|}$$

* Some time, of course, is required for a stable profile to be established.

TABLE I

Enumeration of parameter combinations considered for iron-iron impact. The units are in the gram-centimeter-microsecond system. $\rho = 7.8 \text{ gm/cm}^3$, $D_s = 1$. In computing the required net size for the explicit scheme, a is chosen as 1.5, $t = 1.5 \Delta x$.

No.	Parameters	Bingham No. $B_o = \frac{D_s \tau_o}{\mu_o v_o}$	Reynolds No. $R_o = \frac{D_s \rho v_o}{\mu_o}$	Stability Criteria $\Delta t / (\Delta x)^2$	$\Delta t / \Delta x$	Net Size $\Delta x = 0.1$ Δt
1	0.5 0 0	-	∞	3.06	0.60	6.0×10^{-2}
2	0.5 0 8×10^{-2}	0	4.88×10	2.70		2.7 "
3	0.5 0 8×10^{-1}	0	4.88	1.38		1.4 "
4	0.5 0 8	0	4.88×10^{-1}	0.23		2.3×10^{-3}
5	0.5 10^{-2} 0	∞	∞	3.06	0.60	6.0×10^{-2}
6	0.5 10^{-2} 8×10^{-2}	2.5×10^{-1}	4.88×10	2.70		2.7 "
7	0.5 10^{-2} 8×10^{-1}	2.5×10^{-2}	4.88	1.38		1.4 "
8	0.5 10^{-2} 8	2.5×10^{-3}	4.88×10^{-1}	0.23		2.3×10^{-3}
9	0.5 10^{-1} 0	∞	∞	3.06	0.60	6.0×10^{-1}
10	0.5 10^{-1} 8×10^{-2}	2.5	4.88×10	2.70		2.7 "
11	0.5 10^{-1} 8×10^{-1}	2.5×10^{-1}	4.88	1.38		1.4 "
12	0.5 10^{-1} 8	2.5×10^{-2}	4.88×10^{-1}	0.23		2.3×10^{-3}
13	0.5 1 0	∞	∞	3.06	0.60	6.0×10^{-1}
14	0.5 1 8×10^{-2}	2.5×10	4.88×10	2.70		2.7 "
15	0.5 1 8×10^{-1}	2.5	4.88	1.38		1.4 "
16	0.5 1 8	2.5×10^{-1}	4.88×10^{-1}	0.23		2.3×10^{-3}
17	4 0 0	-	∞	0.21	0.44	2.1 "
18	4 0 8×10^{-2}	0	4.88×10	0.21		2.1 "
19	4 0 8×10^{-1}	0	4.88	0.18		1.8 "
20	4 0 8	0	4.88×10^{-1}	0.085		8.5×10^{-4}
21	4 10^{-2} 0	∞	∞	0.21	0.44	2.1×10^{-3}
22	4 10^{-2} 8×10^{-2}	3.13×10^{-2}	4.88×10	0.21		2.1 "
23	4 10^{-2} 8×10^{-1}	3.13×10^{-3}	4.88	0.18		1.8 "
24	4 10^{-1} 8	3.13×10^{-4}	4.88×10^{-1}	0.085		8.5×10^{-4}
25	4 10^{-1} 0	∞	∞	0.21	0.44	2.1×10^{-1}
26	4 10^{-1} 8×10^{-2}	3.13×10^{-1}	4.88×10	0.21		2.1 "
27	4 10^{-1} 8×10^{-1}	3.13×10^{-2}	4.88	0.18		1.8 "
28	4 10^{-1} 8	3.13×10^{-3}	4.88×10^{-1}	0.085		8.5×10^{-4}
29	4 1 0	∞	∞	0.21	0.44	2.1×10^{-3}
30	4 1 8×10^{-2}	3.13	4.88×10	0.21		2.1 "
31	4 1 8×10^{-1}	3.13×10^{-1}	4.88	0.18		1.8 "
32	4 1 8	3.13×10^{-2}	4.88×10^{-1}	0.085		8.5×10^{-4}
33	7.5 0 0	-	∞	0.093	0.40	9.3 "
34	7.5 0 8×10^{-2}	0	4.88×10	0.092		9.2 "
35	7.5 0 8×10^{-1}	0	4.88	0.086		8.6 "
36	7.5 0 8	0	4.88×10^{-1}	0.051		5.1 "
37	7.5 10^{-2} 0	∞	∞	0.093	0.40	9.3 "
38	7.5 10^{-2} 8×10^{-2}	1.67×10^{-2}	4.88×10	0.092		9.2 "
39	7.5 10^{-2} 8×10^{-1}	1.67×10^{-3}	4.88	0.086		8.6 "
40	7.5 10^{-1} 8	1.67×10^{-4}	4.88×10^{-1}	0.051		5.1 "
41	7.5 10^{-1} 0	∞	∞	0.093	0.40	9.3 "
42	7.5 10^{-1} 8×10^{-2}	1.67×10^{-1}	4.88×10	0.092		9.2 "
43	7.5 10^{-1} 8×10^{-1}	1.67×10^{-2}	4.88	0.086		8.6 "
44	7.5 10^{-1} 8	1.67×10^{-3}	4.88×10^{-1}	0.051		5.1 "
45	7.5 1 0	∞	∞	0.093	0.40	9.3 "
46	7.5 1 8×10^{-2}	1.67	4.88×10	0.092		9.2 "
47	7.5 1 8×10^{-1}	1.67×10^{-1}	4.88	0.086		8.6 "
48	7.5 1 8	1.67×10^{-2}	4.88×10^{-1}	0.051		5.1 "

$$\tau^2 \geq \tau_0^2 \quad \tau^2 = \frac{4}{3} \mu^2 \frac{\rho^2}{\rho_0^2} \left(\frac{\partial q}{\partial x} \right)^2$$

and hence

$$\tau^2 = \tau_0^2 + \frac{4}{\sqrt{3}} \mu_0 \tau_0 \frac{\rho}{\rho_0} \left| \frac{\partial q}{\partial x} \right| + \frac{4}{3} \mu_0^2 \left(\frac{\rho}{\rho_0} \right)^2 \left(\frac{\partial q}{\partial x} \right)^2$$

In front of the disturbance $\partial q/\partial x=0$, in the disturbance $\partial q/\partial x < 0$ and, finally, $\partial q/\partial x$ drops back to zero after the disturbance passes. Consequently, $\tau^2 > \tau_0^2$ in the disturbance, but τ^2 drops to τ_0^2 after it passes. This means, in terms of our visco-plastic model, that there is flow only in that part of the medium through which the disturbance is currently passing; it again becomes rigid behind the disturbance.

EXPLICIT DIFFERENCE SCHEME

(a) Stability Criteria.

In the Second Quarterly Report a trial stability criterion was suggested for the finite difference scheme presented there. It was deduced by analogy from a criterion derived for the perfect fluid equations. In performing preliminary machine calculations the trial criterion was found to fail for the higher values of the impact velocity and viscosity parameters. Therefore, this phenomenon had to be investigated more thoroughly. In the Appendix the stability analysis is presented in some detail; the essential results are contained in (A-22) and (A-24).

To apply criterion (A-22) we recall that

$$\frac{\partial V}{\partial t} = \frac{1}{\rho_0} \frac{\partial q}{\partial x}$$

and approximate $\partial q/\partial x$ by $-v_0/2\Delta x$. Here $v_0/2$ represents the velocity of the interface as calculated from the Rankine-Hugoniot conditions (Ref. 2 eq. 64). In our calculations we have chosen $\beta = a \Delta x$, $a = 1.5$, so that the criterion reduces to

$$(1) \quad \frac{\Delta t}{(\Delta x)^2} \leq \frac{\rho_0^2 V}{\frac{8}{3} \mu_0 + 4.5 \rho_0 v_0 \Delta x}$$

The value for $V = 1/\rho$ should be taken from the Hugoniot curve for the particular material (Ref. 2, Fig. 4).

Now, if we are away from the shock $p \gg Q + \sqrt{\frac{4}{3}} \tau_0$ and condition (A-24) becomes roughly

$$\frac{\Delta t}{\Delta x} \leq \frac{\rho_0}{2 \sqrt{2 \left| \frac{\partial g}{\partial V} - p \frac{\partial g}{\partial U} \right|}}$$

The adiabatic sound speed c is equal to ²

$$c = \sqrt{\left| \frac{\partial g}{\partial V} - p \frac{\partial g}{\partial U} \right|} = c_0 \sqrt{\rho_0 V}$$

Consequently, in regions other than those containing a shock,

$$(2) \quad \frac{\Delta t}{\Delta x} \leq \sqrt{\frac{\rho_0 V}{8 c_0^2}} \quad (\mu_0 = 0)$$

is the approximate stability criterion. This is the tentative requirement for stability suggested in the Second Quarterly Report (Ref. 2, eq. 66).

If $(V_0/2) > c$, a shock will occur in the impacting materials and criterion (1) must be applied even when $\mu_0 = 0$. In the case of iron, for example, $c \approx 0.5 \text{ cm/microsec.}$ and, consequently, criterion (2) holds when $V_0 = 0.5$, while at impact velocities 4 and 7.5 criterion (1) obtains. The stability criteria have been calculated for each of the parameter combinations considered. The results are listed in Table I along with the corresponding value of Δt for the choice $\Delta x = 0.1 \text{ cm}$.

The above conclusions have been substantiated by a number of machine calculations, the results are displayed in Table II. It is seen that the calculated requirement on Δt is a conservative estimate in each case. It may be expected that the theoretical stability criterion is close to the required condition for all the

²In an adiabatic process $dU + p dV = 0$. Since $p = g(V, U)$ we may write $dp = dV \frac{\partial g}{\partial V} + dU \frac{\partial g}{\partial U}$. Elimination of dU between the two yields $dp = (\frac{\partial g}{\partial V} - p \frac{\partial g}{\partial U}) dV = V^2 (p \frac{\partial g}{\partial U} - \frac{\partial g}{\partial V}) dp$, whence

$$c^2 = dp/d\rho = V^2 (p \frac{\partial g}{\partial U} - \frac{\partial g}{\partial V}).$$

TABLE II

Summary of preliminary runs testing stability of explicit difference scheme applied to iron-iron impact. $a = 1.5$, $\ell = 1.5 \Delta x$, $\Delta x = 0.1$.

NO.	NET SIZE		RESULT
	Theory	Trial	
1	0.06	0.100	Unstable
"	"	0.050	Stable
"	"	0.065	Stable
2	0.027	0.05	Stable
3	0.014	0.02	Stable
4	0.0023	0.005	Unstable
"	"	0.0025	Unstable
"	"	0.002	Stable
13	0.06	0.05	Stable
17	0.0021	0.05	Unstable
"	"	0.025	Unstable
"	"	0.01	Unstable
"	"	0.005	Stable

parameter combinations. For the most severe case, $\Delta t = 5.1 \times 10^{-4}$, this means that approximately 10,000 cycles are required for a 5 microsec. run. If 70 space mesh points are required this means 700,000 point calculations which would require approximately 1.4 hours on the IBM 7090. An inordinate amount of machine time might therefore be needed to complete the desired production runs. An alternate implicit difference scheme is therefore proposed for those parameter combinations with larger μ_0 and v_0 values. Before turning to this, however, let us first examine the results of the calculations which have already been made using the explicit scheme.

(b) Results of Calculations

To check the accuracy of the program the calculations for case No. 1 ($v_0 = 0.5$, $\tau_0 = 0$, $\mu_0 = 0$) have been examined in detail. The pressure profiles at various time intervals are depicted in Fig. 1. Since $\tau_0 = \mu_0 = 0$ the Rankine-Hugoniot solution holds and, for comparison, these exact pressure values and shock positions are shown at $t = 1$ and 5 microsec. It is seen that the shock is better approximated with increasing time, but only 2 to 3 microsec. are required for sufficient accuracy.

The other calculated dependent variables may also be compared with the corresponding Rankine-Hugoniot values for case No. 1:

	$\frac{v}{v_0}$	U	p	q
HUGONIOT	0.08730	0.03125	1.565	0.2500
CALCULATED	0.0867	0.027	1.54	0.25

The calculated quantities represent mean values about which there are small oscillations at the various space mesh points behind the disturbance. The largest percentage discrepancy is in the specific internal energy.

The location where p is half its maximum value is considered to be the shock position. At each time cycle this criterion is used to compute the shock velocity in that time increment. In Fig. 2 the calculated shock velocities (relative to the interface) for case No. 1 are shown. As these are found to oscillate widely, an average shock velocity is computed according to the formula:

$$\text{Shock Velocity} = \frac{\text{Position of Shock} - \text{Position of Interface}}{\text{Time}}$$

It is seen in Fig. 2 that the shock velocity computed in this fashion converges more rapidly and more smoothly to the Rankine-Hugoniot solution. The ratio of this value to the true shock velocity for various times is given by

Time (microsec.)	1	2	3	4	5
Ratio	1.100	1.034	1.015	1.005	0.999

In Figs. 3, 4 the disturbance (pulse) velocities calculated in this manner are depicted as a function of time. Each is seen to be tending smoothly towards an asymptotic value represented by a dashed line. The value of the asymptote was computed under the rough assumption that the rate of approach at corresponding intervals is the same when $|\mu_0| + |\tau_0| \neq 0$ as it was for Case 1, i.e., the ratios displayed above for the perfect fluid calculations are employed in estimating all the asymptotic values.

In Fig. 5 the value of τ^2 is shown at various time intervals for a typical parameter combination (No. 14). It is seen that at each instant $\tau^2 > \tau_0^2$ only in a small region which represents the current position of the disturbance. It is only in this moving region of disturbance that the medium behaves as a viscous liquid.

A summary of all the calculations which have been performed are presented in Table III.

TABLE III

Summary of exploratory calculations made with the explicit difference scheme. $a = 1.5$, $\Delta x = 0.1$ cm, $\Delta t = 0.05 \mu$ sec. Values represent averages between the interface and the position of the disturbance after 5 microseconds. The units are in the gram-centimeter-microsecond system.

No.	$r_0 = 0.5$		P	U	V($= 1/\rho$)	ρ/ρ_0	Disturbance Velocity
	τ_0	μ_0					
1	0	0	1.538	0.0265	0.0868	1.466	0.547
2	0	0.08	1.550	0.0293	0.0871	1.461	0.549
9	0.1	0	1.567	0.0327	0.0876	1.452	0.560
10	0.1	0.08	1.555	0.0324	0.0877	1.450	0.562
14	1.0	0.08	1.737	0.0681	0.0897	1.418	0.661

IMPLICIT DIFFERENCE SCHEME

In this section an alternate computational scheme is outlined which is devised to reduce the machine time required for those parameter combinations with large μ_0 and v_0 values. It is an implicit scheme based on the Eulerian formulation, whereas the original scheme is explicit and is based on the Lagrangian formulation. The new scheme is only valid for impact between bodies of identical material with $\mu_0 \neq 0$. Therefore it does not supersede the explicit scheme, but serves as a desirable complement.

The procedure has been adopted upon the suggestion of Dr. Herbert Keller, Institute of Mathematical Sciences, New York University. In treating similar systems of equations, Dr. Keller has found the implicit scheme to be unconditionally stable. Thus, no restriction on Δt and Δx is involved, only the desired accuracy need be considered in choosing the increment sizes.

(a) Difference Equations

Since the two impacting bodies are identical the phenomena are symmetric about the center of mass coordinates (Fig. 6). The calculations will be made only for body 2 where the fixed space coordinates are denoted by $j = 0, 1, 2, \dots$. There $\partial q / \partial z \neq 0$ and, consequently, the Eulerian formulations of the governing equations, equations (10) through (21) of Ref. 2, reduce to

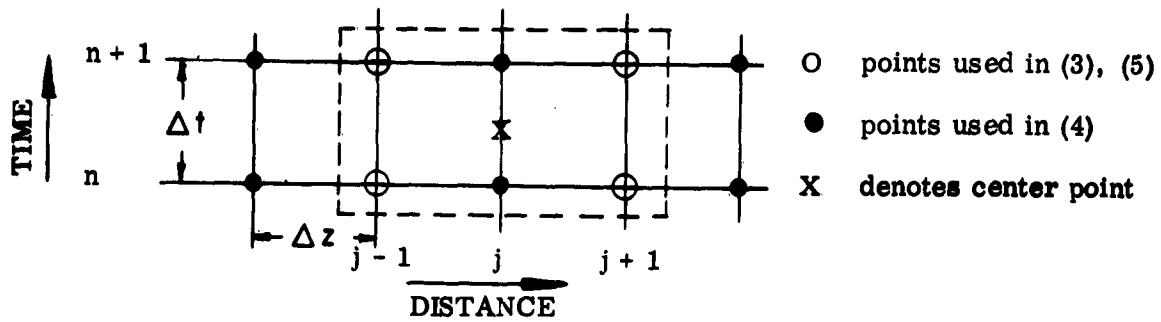
$$(3) \quad \frac{\partial \rho}{\partial t} + \frac{\partial(\rho q)}{\partial z} = 0$$

$$(4) \quad \frac{\partial q}{\partial t} + q \frac{\partial q}{\partial z} = \frac{1}{\rho} \left[- \frac{\partial p}{\partial z} + \frac{4}{3} \mu_0 \frac{\partial^2 q}{\partial z^2} \right]$$

$$(5) \quad \frac{\partial U}{\partial t} + q \frac{\partial U}{\partial z} = \frac{1}{\rho} \frac{\partial q}{\partial z} \left[- p + \frac{4}{3} \mu_0 \frac{\partial q}{\partial z} - \sqrt{\frac{4}{3}} \tau_0 \right]$$

$$(6) \quad p = f(\rho, U)$$

A centered difference scheme is used to provide more accuracy. To illustrate the process the details will be carried out for equation (3). The scheme is centered at point $z = j \Delta z$, $t = (n + 1/2) \Delta t$ as shown



in the sketch. Equation (3) is replaced by

$$\frac{1}{2} \left\{ \frac{\rho_{j-1}^{n+1} - \rho_{j-1}^n}{\Delta t} + \frac{\rho_{j+1}^{n+1} - \rho_{j+1}^n}{\Delta t} \right\} + \frac{1}{2} \left\{ \frac{(\rho q)_{j+1}^{n+1} - (\rho q)_{j-1}^{n+1}}{2\Delta z} + \frac{(\rho q)_{j+1}^n - (\rho q)_{j-1}^n}{2\Delta z} \right\} = 0$$

Similar centered schemes may be written for (4) and (5). Upon simplification the system becomes

$$(7) \quad \begin{aligned} \rho_{j-1}^{n+1} + \rho_{j+1}^{n+1} &= \rho_{j-1}^n + \rho_{j+1}^n \\ &+ \frac{1}{2} \frac{\Delta t}{\Delta z} \left\{ \left[(\rho q)_{j-1}^{n+1} + (\rho q)_{j-1}^n \right] - \left[(\rho q)_{j+1}^{n+1} + (\rho q)_{j+1}^n \right] \right\} \end{aligned}$$

$$q_j^{n+1} - q_j^n + \frac{1}{8} \frac{\Delta t}{\Delta s} (q_j^{n+1} + q_j^n) [q_{j+1}^{n+1} - q_{j-1}^{n+1} + q_{j+1}^n - q_{j-1}^n]$$

$$(8) \quad = \frac{1}{2} \frac{\Delta t}{\Delta s} \frac{1}{\rho_j^{n+1} + \rho_j^n} \left\{ [p_{j+1}^{n+1} - p_{j-1}^{n+1} + p_{j+1}^n - p_{j-1}^n] \right. \\ \left. + \frac{8}{3} \frac{\mu_o}{\Delta s} [q_{j+1}^{n+1} - 2q_j^{n+1} + q_{j-1}^{n+1} + q_{j+1}^n - 2q_j^n + q_{j-1}^n] \right\}$$

$$U_{j-1}^{n+1} + U_{j+1}^{n+1} - [U_{j-1}^n + U_{j+1}^n]$$

$$(9) \quad + \frac{1}{8} \frac{\Delta t}{\Delta s} [q_{j-1}^{n+1} + q_{j+1}^{n+1} + q_{j+1}^n + q_{j-1}^n] [U_{j+1}^{n+1} - U_{j-1}^{n+1} + U_{j+1}^n - U_{j-1}^n] \\ = \frac{\Delta t}{\Delta s} \frac{q_{j+1}^{n+1} - q_{j-1}^{n+1} + q_{j+1}^n - q_{j-1}^n}{\rho_{j-1}^{n+1} + \rho_{j+1}^{n+1} + \rho_{j+1}^n + \rho_{j-1}^n} \left\{ -2\sqrt{\frac{4}{3}} \tau_o \right. \\ \left. - \frac{1}{2} [p_{j-1}^{n+1} + p_{j+1}^{n+1} + p_{j+1}^n + p_{j-1}^n] + \frac{2}{3} \frac{\mu_o}{\Delta s} [q_{j+1}^{n+1} - q_{j-1}^{n+1} + q_{j+1}^n - q_{j-1}^n] \right\}$$

The required pressure values are computed from (6) according to

$$(10) \quad p_{j-1}^{n+1} = f(\rho_{j-1}^{n+1}, U_{j-1}^{n+1})$$

$$p_{j+1}^{n+1} = f(\rho_{j+1}^{n+1}, U_{j+1}^{n+1})$$

$$p_{j+1}^n = f(\rho_{j+1}^n, U_{j+1}^n)$$

$$p_{j-1}^n = f(\rho_{j-1}^n, U_{j-1}^n)$$

(b) Initial and Symmetry Conditions

The initial, $n = 0$, values of the dependent variables ρ , q , U , p are all known from the Rankine-Hugoniot equations (Fig. 7). To determine their values at all other time intervals a method of computing the values at time $n + 1$ from those known at time n must be made available. An iteration technique will be devised for this purpose. The following relations, which follow from the symmetry of the problem, will be utilized:

$$(11) \quad \rho_{-j}^n = \rho_j^n, \quad q_{-j}^n = -q_j^n \quad (\text{hence } q_0^n = 0)$$

$$U_{-j}^n = U_j^n, \quad p_{-j}^n = p_j^n.$$

(c) Iteration Procedure

To start the iteration let

$$(12) \quad \tilde{q}_{j(0)}^{n+1} = q_j^n, \quad \tilde{U}_{j(0)}^{n+1} = U_j^n$$

be zero-order approximations to q_j^{n+1} and U_j^{n+1} respectively. Then the first-order approximations are calculated in the following sequence:³

(a-1) Substitute $q_j^{n+1} = \tilde{q}_{j(0)}^{n+1}$ into (7) and calculate $\tilde{\rho}_{j(1)}^{n+1}$ from the resulting two-term recurrence relation ($n + 1$ fixed, j varied):

$$(13) \quad \begin{aligned} \tilde{\rho}_{j-1(1)}^{n+1} + \tilde{\rho}_{j+1(1)}^{n+1} &= \rho_{j-1}^n + \rho_{j+1}^n + \frac{1}{2} \frac{\Delta t}{\Delta z} \left\{ \left[\tilde{\rho}_{j-1(1)}^{n+1} \tilde{q}_{j-1(0)}^{n+1} + (\rho q)_{j-1}^n \right] \right. \\ &\quad \left. - \frac{1}{2} \frac{\Delta t}{\Delta z} \left[\tilde{\rho}_{j+1(1)}^{n+1} \tilde{q}_{j+1(0)}^{n+1} + (\rho q)_{j+1}^n \right] \right\} \end{aligned}$$

³This procedure is adopted upon the suggestion of Dr. Herbert Keller, Institute of Mathematical Sciences, New York University.

Several methods of performing the calculations are discussed below in section (d).

(b-1) To obtain a trial value for ρ_j^{n+1} use (10) with $\rho_j^{n+1} = \tilde{\rho}_{j(1)}^{n+1}$ and $U_j^{n+1} = \tilde{U}_{j(0)}^{n+1}$:

$$(14) \quad \tilde{\rho}_{j(1')}^{n+1} = f(\tilde{\rho}_{j(1)}^{n+1}, \tilde{U}_{j(0)}^{n+1})$$

(c-1) To obtain a first-order approximation for U_j^{n+1} use (9) with q_j^{n+1} , ρ_j^{n+1} , ρ_j^{n+1} replaced by $\tilde{q}_{j(0)}^{n+1}$, $\tilde{\rho}_{j(1)}^{n+1}$, $\tilde{\rho}_{j(1')}^{n+1}$ respectively:

$$(15) \quad \begin{aligned} & \tilde{U}_{j-1(1)}^{n+1} + \tilde{U}_{j+1(1)}^{n+1} - [U_{j-1}^n + U_{j+1}^n] \\ & + \frac{1}{8} \frac{\Delta t}{\Delta z} \left[\tilde{q}_{j-1(0)}^{n+1} + \tilde{q}_{j+1(0)}^{n+1} + q_{j+1}^n + q_{j-1}^n \right] \left[\tilde{U}_{j+1(1)}^{n+1} - \tilde{U}_{j-1(1)}^{n+1} + U_{j+1}^n - U_{j-1}^n \right] \\ & = \frac{\Delta t}{\Delta z} \frac{\tilde{q}_{j+1(0)}^{n+1} - \tilde{q}_{j-1(0)}^{n+1} + q_{j+1}^n - q_{j-1}^n}{\tilde{\rho}_{j-1(1)}^{n+1} + \tilde{\rho}_{j+1(1)}^{n+1} + \rho_{j+1}^n + \rho_{j-1}^n} \quad \left\{ -2\sqrt{\frac{4}{3}} \tau_0 \right. \\ & \quad \left. - \frac{1}{2} \left[\tilde{\rho}_{j-1(1')}^{n+1} + \tilde{\rho}_{j+1(1')}^{n+1} + \rho_{j+1}^n + \rho_{j-1}^n \right] \right. \\ & \quad \left. + \frac{2}{3} \frac{\mu_0}{\Delta z} \left[\tilde{q}_{j+1(0)}^{n+1} - \tilde{q}_{j-1(0)}^{n+1} + q_{j+1}^n - q_{j-1}^n \right] \right\} \end{aligned}$$

As with ρ , we have here a two-term linear recurrence relation for $\tilde{U}_{j(1)}^{n+1}$. Its solution is also discussed in section (d).

(d-1) A first-order approximation for ρ_j^{n+1} is now computed from (10) with $\rho_j^{n+1} = \tilde{\rho}_{j(1)}$ and $U_j^{n+1} = \tilde{U}_{j(1)}$

$$(16) \quad \tilde{\rho}_{j(1)}^{n+1} = f(\tilde{\rho}_{j(1)}^{n+1}, \tilde{U}_{j(1)}^{n+1})$$

(e-1) To calculate a first-order approximation for q_j^{n+1} use (8) with
 $\rho_j^{n+1} = \tilde{\rho}_j^{n+1}(1)$, $p_j^{n+1} = \tilde{p}_j^{n+1}(1)$ and replace q_j^{n+1} in the difference expression
for $q \frac{\partial q}{\partial z}$ by $\tilde{q}_j^n(0)$. The latter substitution linearizes the equation for
 $\tilde{q}_j^{n+1}(1)$, giving

$$(17) \quad \begin{aligned} & \tilde{q}_j^{n+1}(1) - q_j^n + \frac{1}{8} \frac{\Delta t}{\Delta z} (\tilde{q}_j^{n+1}(0) + q_j^n) \left[\tilde{q}_{j+1}^{n+1}(0) - \tilde{q}_{j-1}^{n+1}(0) + q_{j+1}^n - q_{j-1}^n \right] \\ &= \frac{1}{2} \frac{\Delta t}{\Delta z} \frac{1}{\tilde{\rho}_j^{n+1}(1) + \rho_j^n} \left\{ \left[\tilde{p}_{j+1}^{n+1}(1) - \tilde{p}_{j-1}^{n+1}(1) + p_{j+1}^n - p_{j-1}^n \right] \right. \\ & \quad \left. + \frac{8}{3} \frac{\mu_0}{\Delta z} \left[\tilde{q}_{j+1}^{n+1}(1) - 2\tilde{q}_j^{n+1}(1) + \tilde{q}_{j-1}^{n+1}(1) + q_{j+1}^n - 2q_j^n + q_{j-1}^n \right] \right\} \end{aligned}$$

This is a three-term linear recurrence relation for $\tilde{q}_j^{n+1}(1)$ ($n+1$ fixed, j varied); a method of solution is discussed below in section (e).

The second-order approximations are calculated by merely repeating the iteration process: (a-2), (b-2), ..., (e-2). The resulting equations differ from the corresponding equations of the first iteration only in that the subscripts (1), (2) replace the subscripts (0), (1) respectively.

In general, to proceed from the k -order approximation to the $(k+1)$ -order approximation we go through the above iteration process (with subscripts (0), (1) replaced by k and $k+1$ respectively). The process is repeated until a reasonable convergence criterion is satisfied. Usually, only a few cycles are required in such schemes.

Assume that it has been decided that K iterations are sufficient. Then set

$$\rho_j^{n+1} = \tilde{\rho}_j^{n+1}(K), \quad q_j^{n+1} = \tilde{q}_j^{n+1}(K), \quad U_j^{n+1} = \tilde{U}_j^{n+1}(K), \quad p_j^{n+1} = \tilde{p}_j^{n+1}(K),$$

and proceed to the $n+2$ time step. A flow chart describing the numerical procedure is given in Fig. 8.

(d) Two-Term Recurrence Relations

It is a consequence of the symmetry relations (11) that when $j=0$ the two-term recurrence relation (a-1) simplifies and yields the explicit formula

$$(18) \quad \tilde{\rho}_{1(1)}^{n+1} = \rho_1^n \frac{1 - \frac{1}{2} \frac{\Delta t}{\Delta z} q_1^n}{1 + \frac{1}{2} \frac{\Delta t}{\Delta z} \tilde{q}_{1(0)}^{n+1}}$$

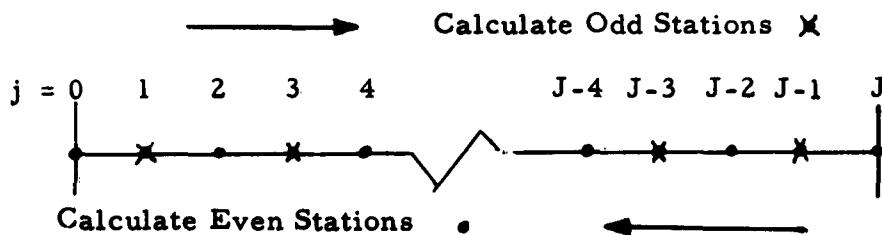
Now (a-1) may be used to compute $\tilde{\rho}_{j(1)}^{n+1}$ at all the odd j space-mesh points:

$$(19) \quad \begin{aligned} & \tilde{\rho}_{j+2(1)}^{n+1} \left[1 + \frac{1}{2} \frac{\Delta t}{\Delta z} \tilde{q}_{j+2(0)}^{n+1} \right] + \tilde{\rho}_{j(1)}^{n+1} \left[1 - \frac{1}{2} \frac{\Delta t}{\Delta z} q_{j(1)}^{n+1} \right] \\ & = \rho_{j+2}^n \left[1 - \frac{1}{2} \frac{\Delta t}{\Delta z} q_{j+2}^n \right] + \rho_j^n \left[1 + \frac{1}{2} \frac{\Delta t}{\Delta z} q_j^n \right] \end{aligned}$$

Thus, set $j = 1$ and compute $\tilde{\rho}_{3(1)}^{n+1}$ in terms of $\tilde{\rho}_{1(1)}^{n+1}$, set $j = 3$ and compute $\tilde{\rho}_{5(1)}^{n+1}$ in terms of $\tilde{\rho}_{3(1)}^{n+1}$, etc.

The $\tilde{\rho}_{j(1)}^{n+1}$ at even j are determined by the continuity of ρ — for sufficiently large j , $\rho_j^{n+1} = \rho_0$ (the density of the undisturbed medium, which has not yet been reached by the shock wave). Let J denote such a large even integer at time $t = (n+1) \Delta t$. Then $\tilde{\rho}_{J(1)}^{n+1} = \rho_0$ and the recursion formula is used to calculate from right to left. Thus, set $j = J-2$ and compute $\tilde{\rho}_{J-2(1)}^{n+1}$ in terms of $\tilde{\rho}_{J(1)}^{n+1}$, set $j = J-4$ and compute $\tilde{\rho}_{J-4(1)}^{n+1}$ in terms of $\tilde{\rho}_{J-2(1)}^{n+1}$, etc.

A suitable value for J may be found by first making the calculations for the odd valued mesh points until $\tilde{\rho}_{j(1)}^{n+1}$ has decreased to the value ρ_0 . This value is then taken to be $J-1$. This left-to-right-to-left technique is depicted in the sketch.



Alternatively, the choice for J may be made from a knowledge of the propagation velocity of the disturbance. Then the calculations for both the odd and the even stations may be performed from right-to-left. The advantage of this procedure is that (18) may then be reserved to check the calculations.

The remarks made above are also applicable to the two-term linear recurrence relation for \tilde{U}_j^{n+1} . Here the symmetry relations together with (15), with $j=0$, yield the explicit equation

$$(20) \quad \begin{aligned} \tilde{U}_1^{n+1} = U_1^n + \frac{\Delta t}{\Delta z} \frac{\tilde{q}_1^{n+1}(0) + q_1^n}{\tilde{\rho}_1^{n+1}(1) + \rho_1^n} & \left\{ -\sqrt{\frac{4}{3}} \tau_0 \right. \\ & \left. - \frac{1}{2} \left[\tilde{p}_1^{n+1}(1') + p_1^n \right] + \frac{2}{3} \frac{\mu_0}{\Delta z} \left[\tilde{q}_1^{n+1}(0) + q_1^n \right] \right\} \end{aligned}$$

By replacing the word "density" by "specific internal energy" the remainder of the discussion on solving the recurrence relation also carries over.

(e) Three-Term Recurrence Relation

Relation (17) may be rewritten in the form

$$(21) \quad -A_j \tilde{q}_{j+1}^{n+1} + B_j \tilde{q}_j^{n+1} - C_j \tilde{q}_{j-1}^{n+1} = D_j$$

where A_j, B_j, C_j, D_j are known quantities:

$$A_j = \frac{4}{3} \frac{\mu_0}{(\Delta z)^2} \frac{1}{\tilde{\rho}_j^{n+1}(1) + \rho_j^n}$$

$$(22) \quad B_j = 1 + 2A_j \quad C_j = A_j$$

$$\begin{aligned}
 D_j = q_j^n - \frac{1}{8} \frac{\Delta t}{\Delta z} & \left(\tilde{q}_{j-1}^{n+1} + q_j^n \right) \quad \left[\tilde{q}_{j+1}^{n+1} - \tilde{q}_{j-1}^{n+1} + q_{j+1}^n - q_{j-1}^n \right] \\
 (22) \quad & + \frac{1}{2} \frac{\Delta t}{\Delta z} \quad \frac{1}{\tilde{p}_{j+1}^{n+1} + p_j^n} \quad \left\{ \left[\tilde{p}_{j+1}^{n+1} - \tilde{p}_{j-1}^{n+1} + p_{j+1}^n - p_{j-1}^n \right] \right. \\
 & \left. + \frac{8}{3} \frac{\mu_0}{\Delta z} \quad \left[q_{j+1}^n - 2q_j^n + q_{j-1}^n \right] \right\}
 \end{aligned}$$

A method for solving such a three-term linear recurrence relation which is particularly suitable for machine calculations is taken from Richtmyer (Ref. 6, p. 101).

The required inequalities $A_j > 0$, $B_j > 0$, $C_j > 0$ and $B_j > A_j + C_j$ are all seen to be satisfied. The only other requirement is that q_j^{n+1} be specified at the left and right hand boundaries of the space mesh, $j=0$ and $j=J$:

$$(23) \quad \tilde{q}_{0(1)}^{n+1} = 0 \quad \tilde{q}_{J(1)}^{n+1} = -v_0/2$$

CONCLUSIONS

Preliminary runs with the program utilizing the explicit difference scheme have shown that excellent results may be obtained provided the parameters μ_0 and v_0 are not too large. The stability analysis provides estimates, depicted in Table I, which will serve as a guide in the choice of Δt for the production runs to be made on the IBM 7090 at Eglin Air Force Base. The stack of IBM cards for this program are being transmitted along with this report.

The programming of the complementary implicit difference scheme will be completed. When finalized, it will be sent to Eglin where production runs will be made for those parameter combinations with large μ_0 and v_0 values. The two programs should permit the one-dimensional computations for the entire range of impact parameters desired.

Simultaneously, a finite difference scheme for the visco-plastic equations will be sought for the case of the axisymmetric impact problem. It had been anticipated that the particle-in-cell method developed by Harlow (Ref. 7) for a perfect fluid could be extended to our visco-plastic equations. Some effort was spent in this direction but has currently been suspended. The method is similar to the one-dimensional explicit scheme in that an "artificial viscosity" is introduced and it is explicit. Stability problems can therefore be expected when viscosity is included. An alternative method by Kolsky (Ref. 8) may be extended, but it has given best results when the flow is nearly spherical.

It may therefore be expedient to decompose the problem into two parts as suggested by the study of the governing equations detailed in the Second Quarterly Report:

- 1) An initial stage in which only the inertial terms of the governing visco-plastic model are retained. The medium is then a perfect fluid and the particle-in-cell method applies.
- 2) A secondary flow problem in which the flow has smoothed out to a nearly spherical pattern. Here Kolsky's method should yield good results.

The feasibility of such a decomposition will be studied and a continued effort will be made to devise a computational method valid during the entire cratering process.

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APPENDIX

The explicit difference equations are given by (36) through (42) of the Second Quarterly Report. They are as follows:

$$\begin{aligned}
 q_j^{n+1} &= q_j^n - \frac{\Delta t}{\rho_0 \Delta x} \left[p_{j+1/2}^n - p_{j-1/2}^n + Q_{j+1/2}^n - Q_{j-1/2}^n + S_{j+1/2}^n - S_{j-1/2}^n \right] \\
 v_{j+1/2}^{n+1} &= v_{j+1/2}^n + \frac{\Delta t}{\rho_0 \Delta x} \left[q_{j+1}^{n+1} - q_j^{n+1} \right] \\
 S_{j+1/2}^{n+1} &= -\frac{8}{3} \frac{\mu_0}{\Delta t} \frac{1}{v_{j+1/2}^{n+1} + v_{j+1/2}^n} \left[v_{j+1/2}^{n+1} - v_{j+1/2}^n \right] \\
 U_{j+1/2}^{n+1} &= U_{j+1/2}^n - \left[v_{j+1/2}^{n+1} - v_{j+1/2}^n \right] \left[p_{j+1/2}^n + Q_{j+1/2}^n + S_{j+1/2}^n \right] \\
 (A-1) \quad &\quad - \sqrt{\frac{4}{3}} \tau_0 \left[v_{j+1/2}^{n+1} - v_{j+1/2}^n \right] \\
 Q_{j+1/2}^{n+1} &= \frac{2(\rho_0 \ell)^2}{(\Delta t)^2} \frac{1}{v_{j+1/2}^{n+1} + v_{j+1/2}^n} \left[v_{j+1/2}^{n+1} - v_{j+1/2}^n \right]^2 \\
 p_{j+1/2}^{n+1} &= g(v_{j+1/2}^{n+1}, U_{j+1/2}^{n+1})
 \end{aligned}$$

where we have eliminated X from (36) and (38) to obtain the second equation and the equation of state is rewritten as $p = f(V^{-1}, U) = g(V, U)$. Here ℓ is a parameter with dimension of length which essentially determines the magnitude of the pseudo-viscosity.

The analysis of the stability of this system follows the method outlined by Richtmyer (Reference 6). The equations of first variation of (A-1) will be obtained in which quantities of the second and higher order are dropped. This will give us linear equations for the first order variations \dot{q} , \dot{V} , \dot{S} , \dot{U} , \dot{Q} , \dot{p} (the dot does not indicate time derivatives) in which the zero order quantities appear as coefficients. The equations obtained are

$$q_j^{n+1} = q_j^n - \frac{\Delta t}{\rho_0 \Delta x} [p_{j+1/2}^n - p_{j-1/2}^n + Q_{j+1/2}^n - Q_{j-1/2}^n + S_{j+1/2}^n - S_{j-1/2}^n]$$

$$\dot{v}_{j+1/2}^{n+1} = \dot{v}_{j+1/2}^n + \frac{\Delta t}{\rho_0 \Delta x} [q_{j+1}^{n+1} - q_j^{n+1}]$$

$$S_{j+1/2}^{n+1} = -\frac{4}{3} \frac{\rho_0}{\Delta t} \frac{1}{V} [\dot{v}_{j+1/2}^{n+1} - \dot{v}_{j+1/2}^n]$$

$$(A-2) \quad \begin{aligned} \dot{U}_{j+1/2}^{n+1} &= \dot{U}_{j+1/2}^n - (\rho + Q + S + \sqrt{\frac{4}{3}} \tau_0) [\dot{v}_{j+1/2}^{n+1} - \dot{v}_{j+1/2}^n] \\ Q_{j+1/2}^{n+1} &= -\frac{(\rho_0 \tau)^2}{2V^2} \left(\frac{\partial V}{\partial t} \right)^2 [\dot{v}_{j+1/2}^{n+1} + \dot{v}_{j+1/2}^n] \\ &\quad + \frac{2(\rho_0 \tau)^2}{V \Delta t} \frac{\partial V}{\partial t} [\dot{v}_{j+1/2}^{n+1} - \dot{v}_{j+1/2}^n] \\ p_{j+1/2}^{n+1} &= \frac{\partial g}{\partial V} \dot{v}_{j+1/2}^{n+1} + \frac{\partial g}{\partial U} \dot{U}_{j+1/2}^{n+1} \end{aligned}$$

The zero order quantities are considered constants and, consequently, superscripts and subscripts denoting net points are omitted from them.

The first order quantities are assumed to have the Fourier representations

$$(A-3) \quad \begin{aligned} q_j^n &= \sum_k A_k^n e^{ikx} & \dot{v}_j^n &= \sum_k B_k^n e^{ikx} \\ S_j^n &= \sum_k C_k^n e^{ikx} & \dot{U}_j^n &= \sum_k D_k^n e^{ikx} \\ Q_j^n &= \sum_k E_k^n e^{ikx} & p_j^n &= \sum_k F_k^n e^{ikx} \end{aligned}$$

where $x = j\Delta x$. The von Neuman stability criterion is, essentially, that the coefficients A_k^n, \dots, F_k^n remain bounded as the calculations proceed from $t = n\Delta t$ to $t = (n+1)\Delta t$, $t = (n+2)\Delta t$, etc. To investigate this substitute representations (A-3) into (A-2) and set like harmonics equal to zero to obtain the relations

$$(A-4) \quad A_k^{n+1} = A_k^n - i\beta\Delta t (F_k^n + E_k^n + C_k^n)$$

$$(A-5) \quad B_k^{n+1} = B_k^n + i\beta\Delta t A_k^{n+1}$$

$$(A-6) \quad C_k^{n+1} = -\frac{4}{3} \frac{\mu_0}{V\Delta t} [B_k^{n+1} - B_k^n]$$

$$(A-7) \quad D_k^{n+1} = D_k^n - (p+Q+S + \sqrt{\frac{4}{3}} \tau_0) [B_k^{n+1} - B_k^n]$$

$$(A-8) \quad E_k^{n+1} = -\frac{(\rho_0 \ell)^2}{2V^2} \left(\frac{\partial V}{\partial t} \right)^2 [B_k^{n+1} + B_k^n] + \frac{2(\rho_0 \ell)^2}{V\Delta t} \frac{\partial V}{\partial t} [B_k^{n+1} - B_k^n]$$

$$(A-9) \quad F_k^{n+1} = \frac{\partial g}{\partial V} B_k^{n+1} + \frac{\partial g}{\partial U} D_k^{n+1}$$

where

$$(A-10) \quad \beta = \frac{2 \sin(k \Delta x / 2)}{\rho_0 \Delta x}$$

Relations (A-5), (A-6) may be combined to yield

$$C_k^n = -\frac{4}{3} \frac{\mu_0}{V} i\beta A_k^n$$

This may be substituted into (A-4) to obtain

$$(A-11) \quad A_k^{n+1} = \left(1 - \frac{4}{3} \frac{\mu_0}{V} \beta^2 \Delta t \right) A_k^n - i\beta\Delta t [E_k^n + F_k^n]$$

If (A-11) is substituted into (A-6) the result is

$$(A-12) \quad B_k^{n+1} = B_k^n + i\beta\Delta t \left(1 - \frac{4}{3} \frac{\mu_0}{V} \beta^2 \Delta t \right) A_k^n + \beta^2 (\Delta t)^2 [E_k^n + F_k^n]$$

Now (A-12) may be used to eliminate B_k^{n+1} from (A-7) and (A-8) with the results

$$(A-13) \quad D_k^{n+1} = D_k^n - (p+Q+S + \sqrt{\frac{4}{3}} \tau_0) \left[i\beta\Delta t \left(1 - \frac{4}{3} \frac{\mu_0}{V} \beta^2 \Delta t \right) A_k^n + \beta^2 (\Delta t)^2 (E_k^n + F_k^n) \right]$$

$$(A-14) \quad \begin{aligned} E_k^{n+1} = & -\frac{(\rho_0 \ell)^2}{2V^2} \left(\frac{\partial V}{\partial t} \right)^2 \left[2B_k^n + i\beta \Delta t \left(1 - \frac{4}{3} \frac{\mu_0}{V} \beta^2 \Delta t \right) A_k^n + \beta^2 (\Delta t)^2 (E_k^n + F_k^n) \right] \\ & + \frac{2(\rho_0 \ell)^2}{V \Delta t} \frac{\partial V}{\partial t} \left[i\beta \Delta t \left(1 - \frac{4}{3} \frac{\mu_0}{V} \beta^2 \Delta t \right) A_k^n + \beta^2 (\Delta t)^2 (E_k^n + F_k^n) \right] \end{aligned}$$

Finally, substitution of (A-12), (A-14) into (A-9) yields

$$(A-15) \quad \begin{aligned} F_k^{n+1} = & \frac{\partial g}{\partial V} \left\{ B_k^n + i\beta \Delta t \left(1 - \frac{4}{3} \frac{\mu_0}{V} \beta^2 \Delta t \right) A_k^n + \beta^2 (\Delta t)^2 (E_k^n + F_k^n) \right\} \\ & + \frac{\partial g}{\partial U} \left\{ D_k^n - \left(p + Q + S + \sqrt{\frac{4}{3}} \tau_0 \right) \left[i\beta \Delta t \left(1 - \frac{4}{3} \frac{\mu_0}{V} \beta^2 \Delta t \right) A_k^n + \beta^2 (\Delta t)^2 (E_k^n + F_k^n) \right] \right\} \end{aligned}$$

Upon introduction of the notations

$$(A-16) \quad \begin{aligned} \beta &= \frac{2 \sin(k \Delta x / 2)}{\rho_0 \Delta x} & \gamma &= 1 - \frac{4}{3} \frac{\mu_0}{V} \beta^2 \Delta t \\ \Lambda &= p + Q + S + \sqrt{\frac{4}{3}} \tau_0 & \delta &= \frac{\partial g}{\partial V} - \Lambda \frac{\partial g}{\partial U} \\ \epsilon &= \frac{1}{2V} \frac{\partial V}{\partial t} & \zeta &= \frac{(\rho_0 \ell)^2}{V} \frac{\partial V}{\partial t} \end{aligned}$$

the relations (A-11) through (A-15) may be written in the matrix form

$$(A-17) \quad Y^{n+1} = G Y^n$$

where

$$Y^n = \begin{bmatrix} A_k^n \\ B_k^n \\ D_k^n \\ E_k^n \\ F_k^n \end{bmatrix} \quad Y^{n+1} = \begin{bmatrix} A_k^{n+1} \\ B_k^{n+1} \\ D_k^{n+1} \\ E_k^{n+1} \\ F_k^{n+1} \end{bmatrix}$$

and the "amplification matrix", which depends on μ_o and τ_o , is given by

$$(A-18) \quad G = \begin{bmatrix} \gamma & 0 & 0 & -i\beta\Delta t & -i\beta\Delta t \\ i\beta\gamma\Delta t & 1 & 0 & \beta^2(\Delta t)^2 & \beta^2(\Delta t)^2 \\ -i\beta\gamma\Lambda\Delta t & 0 & 1 & -\beta^2\Lambda(\Delta t)^2 & -\beta^2\Lambda(\Delta t)^2 \\ i\beta\gamma\zeta\Delta t\left(\frac{2}{\Delta t} - \epsilon\right) & -2\epsilon\zeta & 0 & \beta^2\zeta(\Delta t)^2\left(\frac{2}{\Delta t} - \epsilon\right) & \beta^2\zeta(\Delta t)^2\left(\frac{2}{\Delta t} - \epsilon\right) \\ i\beta\gamma\delta\Delta t & \frac{\partial g}{\partial V} & \frac{\partial g}{\partial U} & \beta^2\delta(\Delta t)^2 & \beta^2\delta(\Delta t)^2 \end{bmatrix}$$

Now, expansion of the determinant $|G - \lambda I|$ and setting the result equal to zero shows that the eigenvalues, λ , of G satisfy the equation

$$(A-19) \quad \lambda^2(\lambda-1) \left\{ (\lambda-1)^2 - (\lambda-1)\beta^2\Delta t \left[\delta\Delta t + \zeta(2-\epsilon\Delta t) - \frac{4}{3} \frac{\mu_o}{V} \right] + \beta^2(\Delta t)^2(2\epsilon\zeta - \delta) \right\} = 0$$

If $\ell = \text{constant}$ and $\Delta t/(\Delta x)^2 = O(1)$ as $\Delta t, \Delta x \rightarrow 0$, then $\beta^2\Delta t = O(1)$ and the secular equation reduces to

$$(A-20) \quad \lambda^2(\lambda-1)^2 \left\{ \lambda-1 - \beta^2\Delta t \left[2\zeta - \frac{4}{3} \frac{\mu_o}{V} \right] - O(\Delta t) \right\}$$

Thus, the von Neuman requirement for stability

$$(A-21) \quad |\lambda_m| \leq 1 + O(\Delta t)$$

is satisfied provided

$$\beta^2\Delta t \left[\frac{2}{3} \frac{\mu_o}{V} + \frac{(\rho_o\ell)^2}{V} \left| \frac{\partial V}{\partial t} \right| \right] \leq 1$$

where we have used the fact that $\partial V/\partial t < 0$. The inequality will hold provided

$$(A-22) \quad \frac{\Delta t}{(\Delta x)^2} \leq \frac{\rho_o^2 V}{\frac{8}{3} \mu_o + 4(\rho_o\ell)^2 \left| \frac{\partial V}{\partial t} \right|}$$

It may be noted from (A-22) that in the limit as the viscosity tends to zero the stability criterion reduces to

$$(A-23) \quad \frac{\Delta t}{(\Delta x)^2} \leq \frac{V}{4\ell^2 \left| \frac{\partial V}{\partial t} \right|} \quad (\mu_o = 0)$$

which is the value given by Richtmyer (Ref. 6, p. 220) for this case.

In the case $\mu_0 = 0$ it is also possible to derive a stability criterion from (A-19) and (A-21) under the condition that $\ell = a \Delta x$, where a is some constant, instead of holding ℓ constant. Then the restriction is found to be relaxed to

$$(A-24) \quad \frac{\Delta t}{\Delta x} \leq \frac{\rho_0}{2 \sqrt{2 \left| \frac{\partial g}{\partial V} - \left(p + Q + \frac{4}{3} \tau_0 \right) \frac{\partial g}{\partial U} \right|}}$$

Criterion (A-23) holds in the region of the shock, and criterion (A-24) is valid in regions away from the shock. However, if true viscosity is present ($\mu_0 \neq 0$) then we must always have $\Delta t / (\Delta x)^2 = O(1)$ for stability; if $\Delta t / \Delta x = O(1)$ one of the eigenvalues goes to infinity. This results since in this case the μ_0 term in (A-19) does not have a factor of order $(\Delta x)^2$ but of order unity.

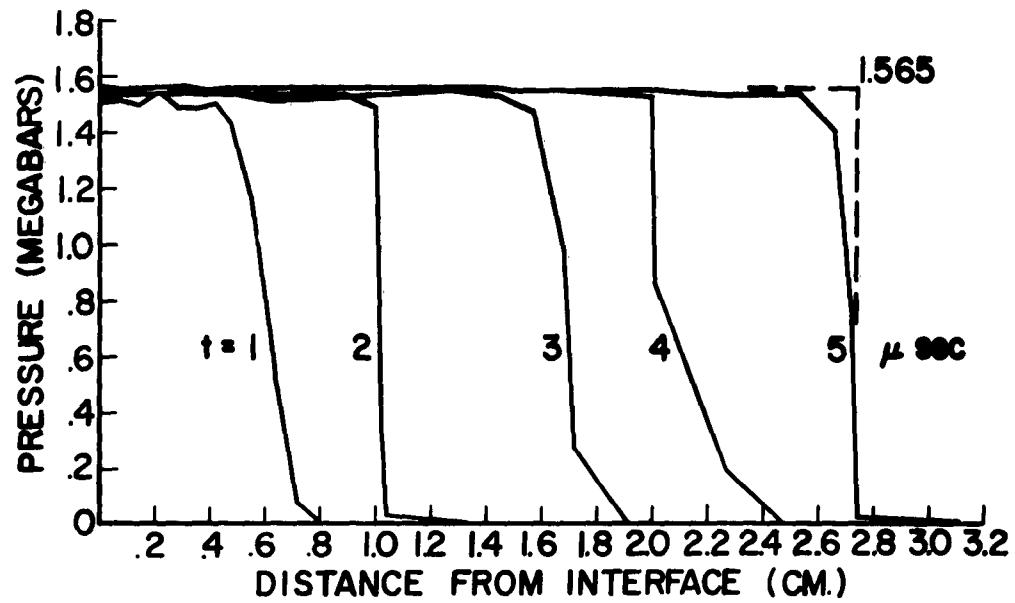
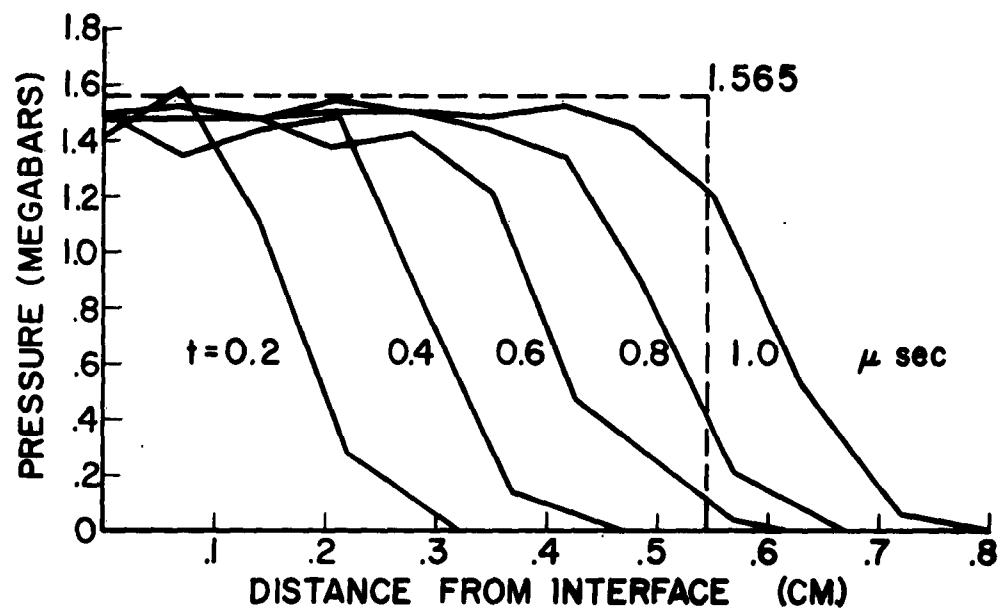


FIG. I CALCULATED PRESSURE PROFILE ($\alpha=1.5$, $\Delta x = 0.1$, $\Delta t = 0.05$) COMPARED TO THE RANKINE-HUGONIOT SOLUTION

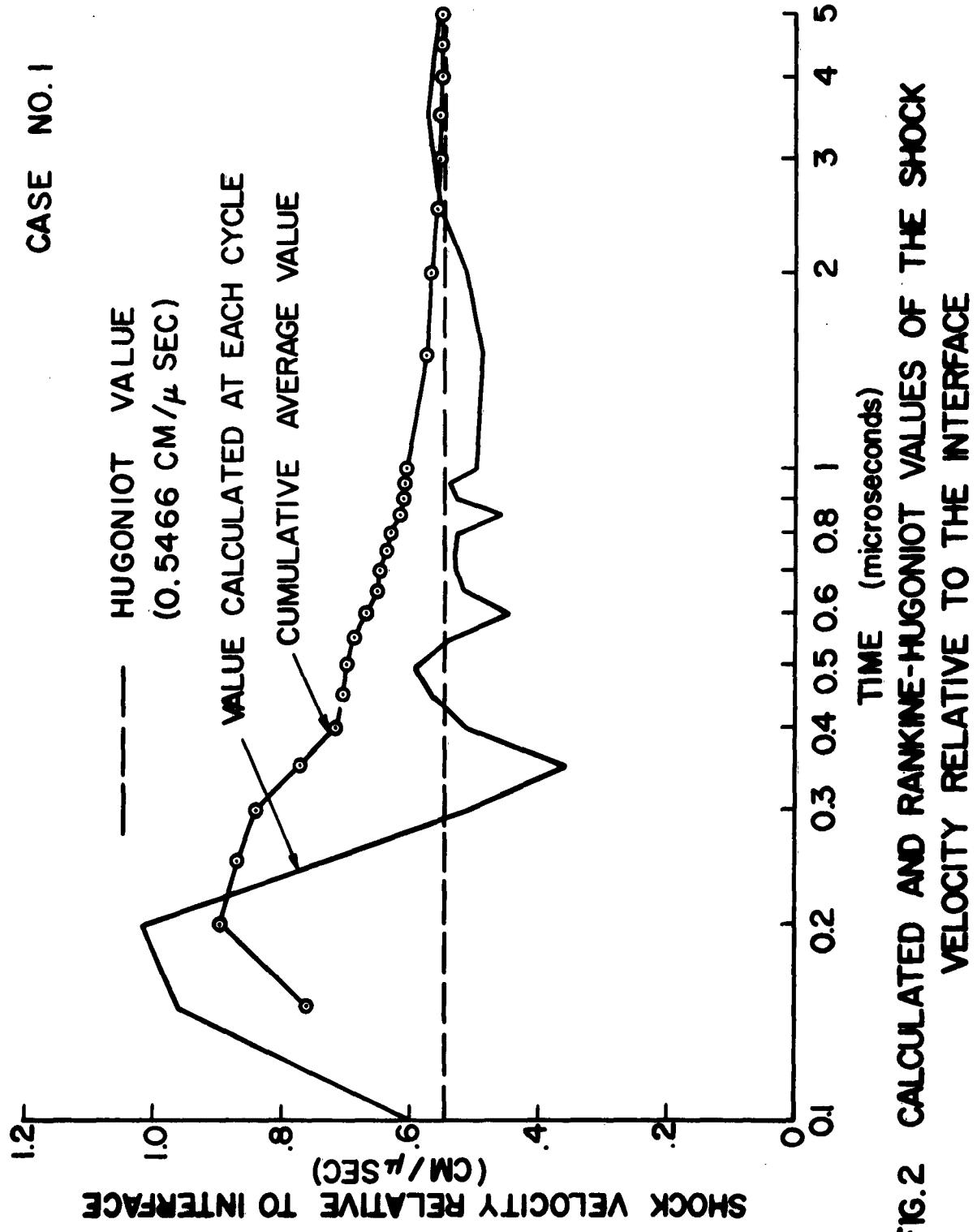


FIG. 2 CALCULATED AND RANKINE-HUGONIOT VALUES OF THE SHOCK VELOCITY RELATIVE TO THE INTERFACE

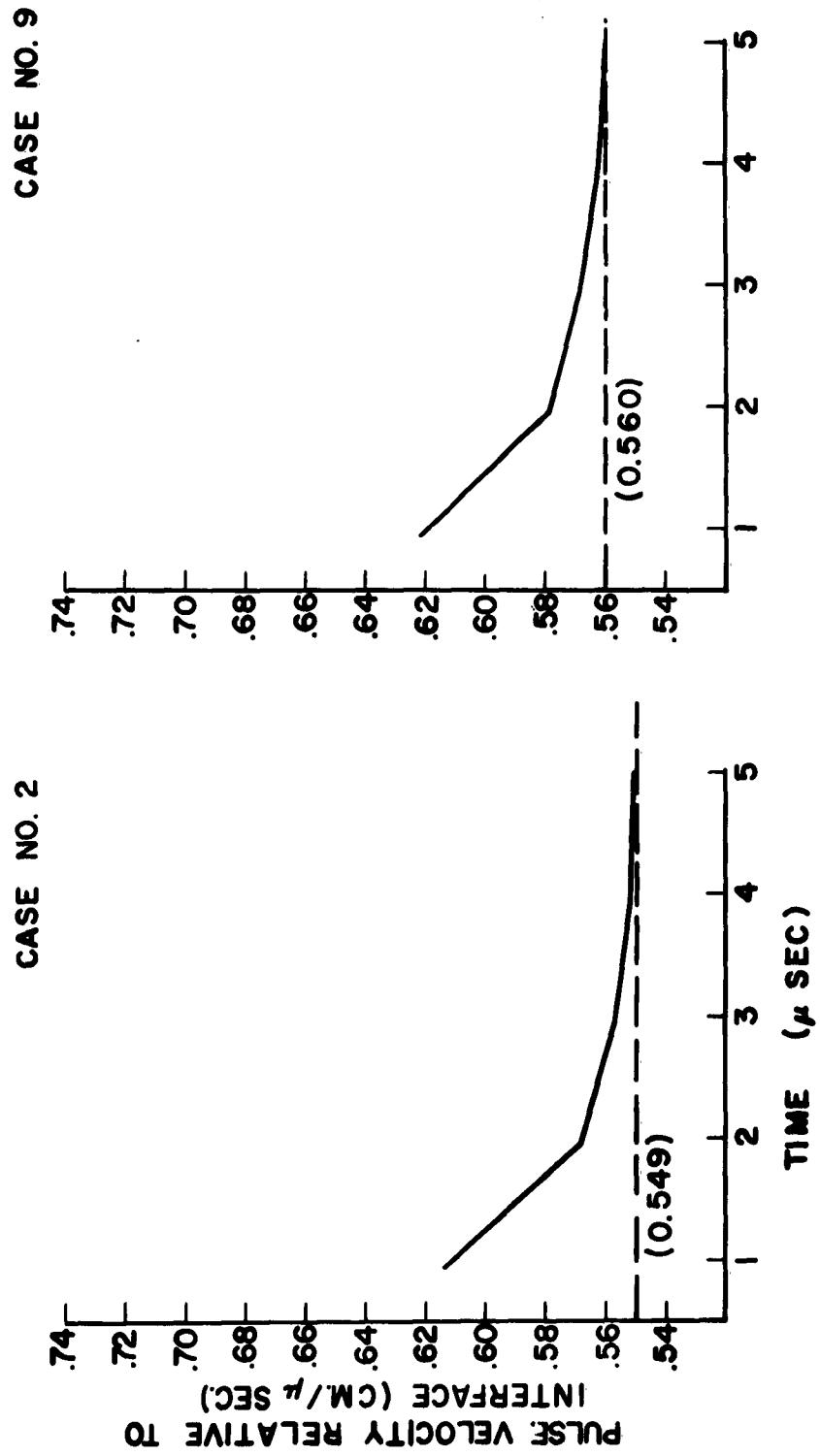


FIG. 3 PULSE VELOCITIES AT VARIOUS TIMES COMPARED WITH ASYMPTOTIC VALUES

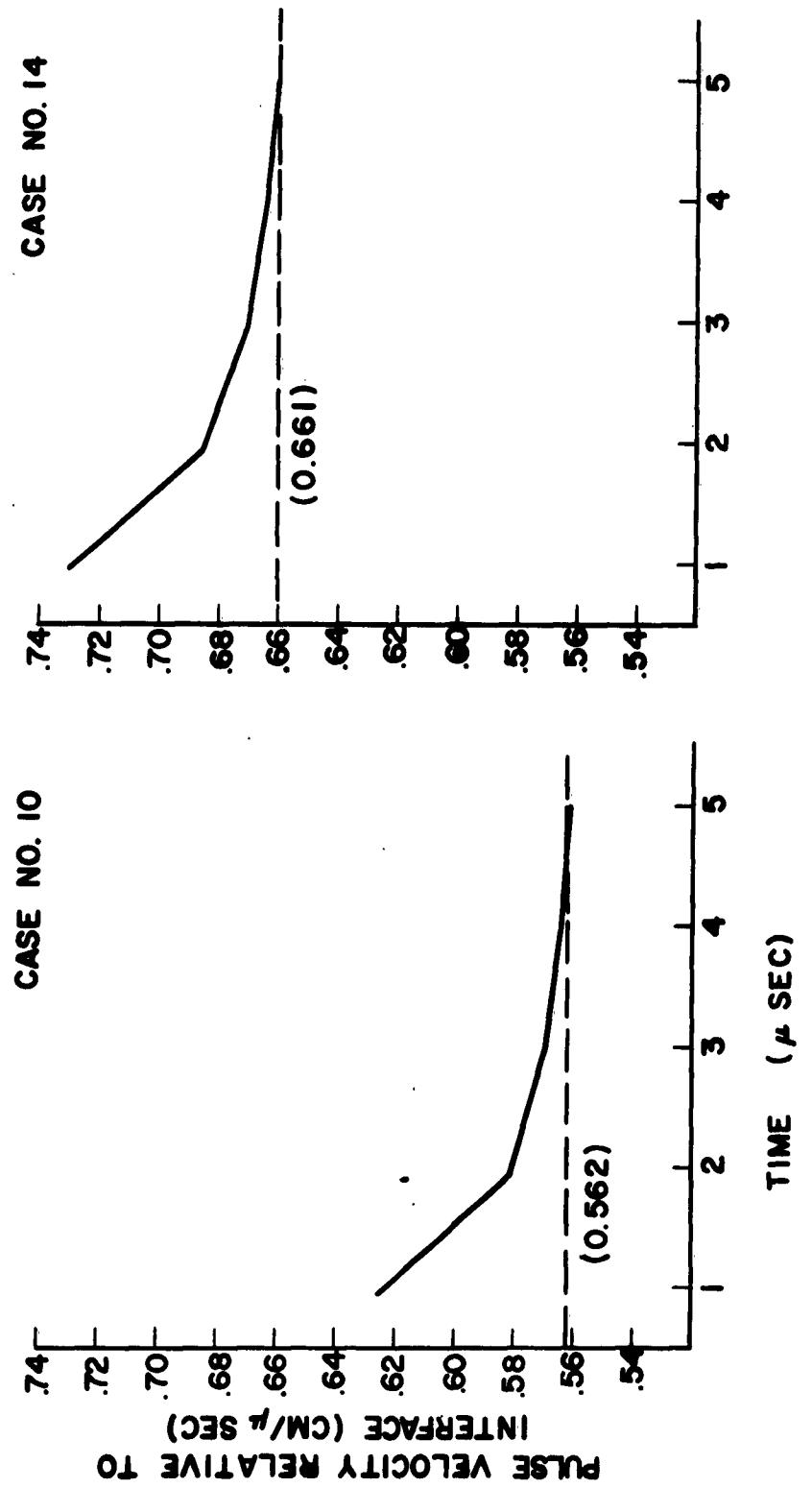


FIG. 4 PULSE VELOCITIES AT VARIOUS TIMES COMPARED
WITH ASYMPTOTIC VALUES

CASE NO. 14

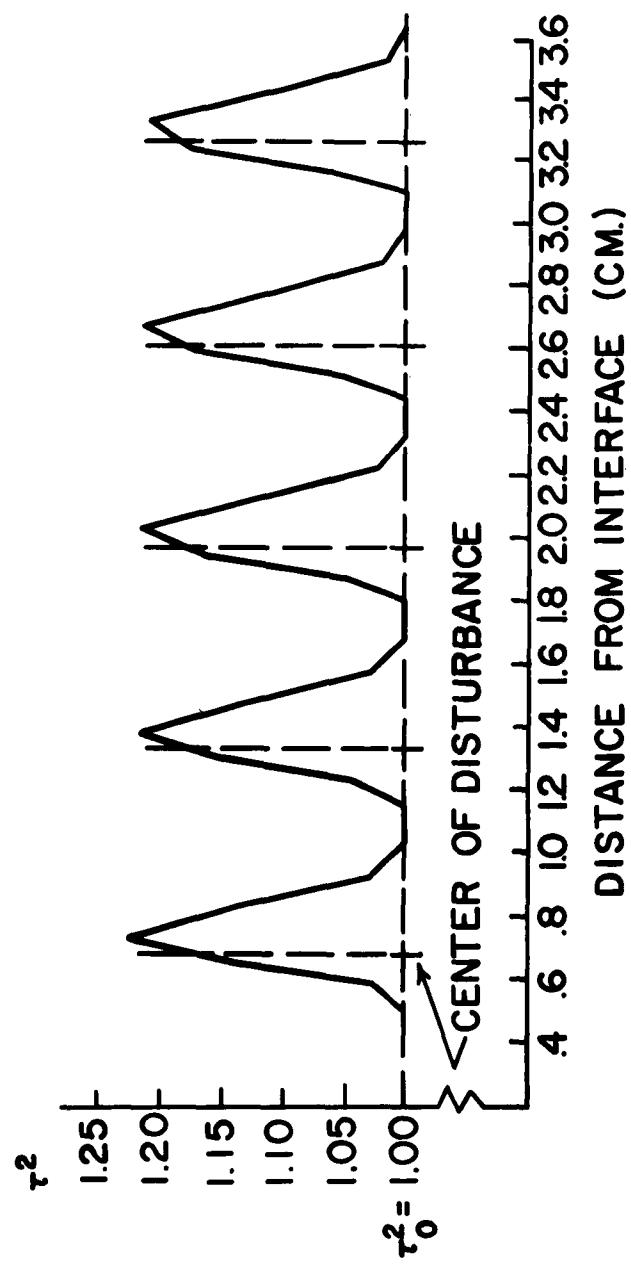
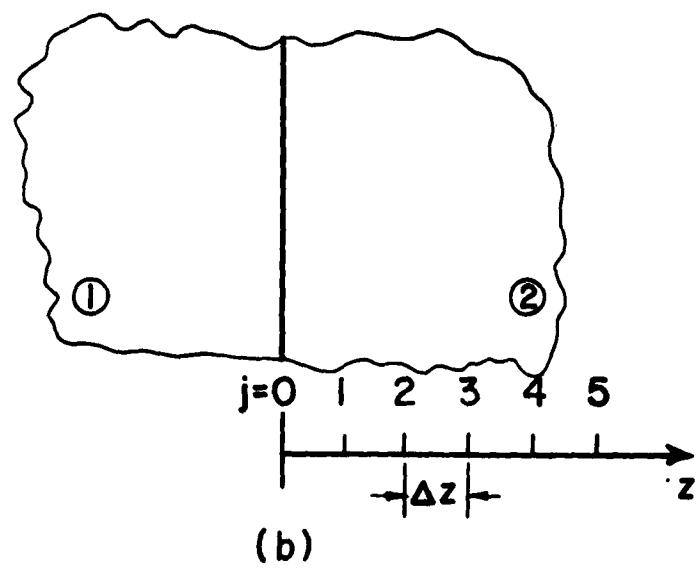
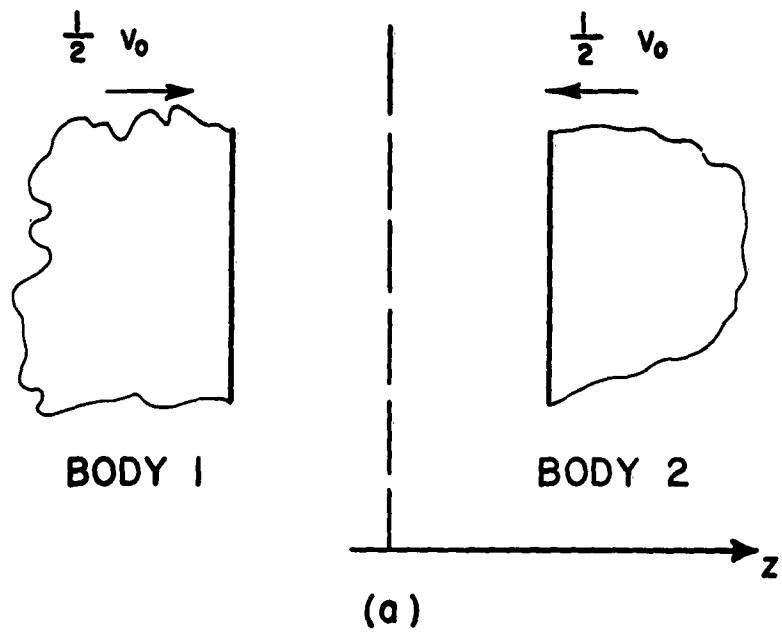


FIG. 5 PLOT OF VON MISES FLOW STATISTIC AT VARIOUS POSITIONS
OF THE DISTURBANCE



**FIG.6 ILLUSTRATION OF IMPACT SITUATION SHOWING
(a) CENTER OF MASS COORDINATES, (b) EULERIAN
SPACE-MESH POINTS**

	BODY 1		BODY 2	
Station \ Quantity	- ∞ to -1	0	0	1 to ∞
q	$v_o/2$	0	0	$-v_o/2$
ρ	ρ_o	$\rho_o(1-\rho_o/d)^{-1}$	$\rho_o(1-\rho_o/d)^{-1}$	ρ_o
U	0	$v_o^2/8$	$v_o^2/8$	0
p	0	$v_o^2 d/4$	$v_o^2 d/4$	0

Here d is determined by the relation (Ref. 2, eq. 63)

$$\frac{v_o^2}{4} d = f\left(\frac{\rho_o}{1-\rho_o/d}, \frac{v_o^2}{8}\right)$$

Fig. 7 Tabular display of initial and boundary data for implicit difference scheme.

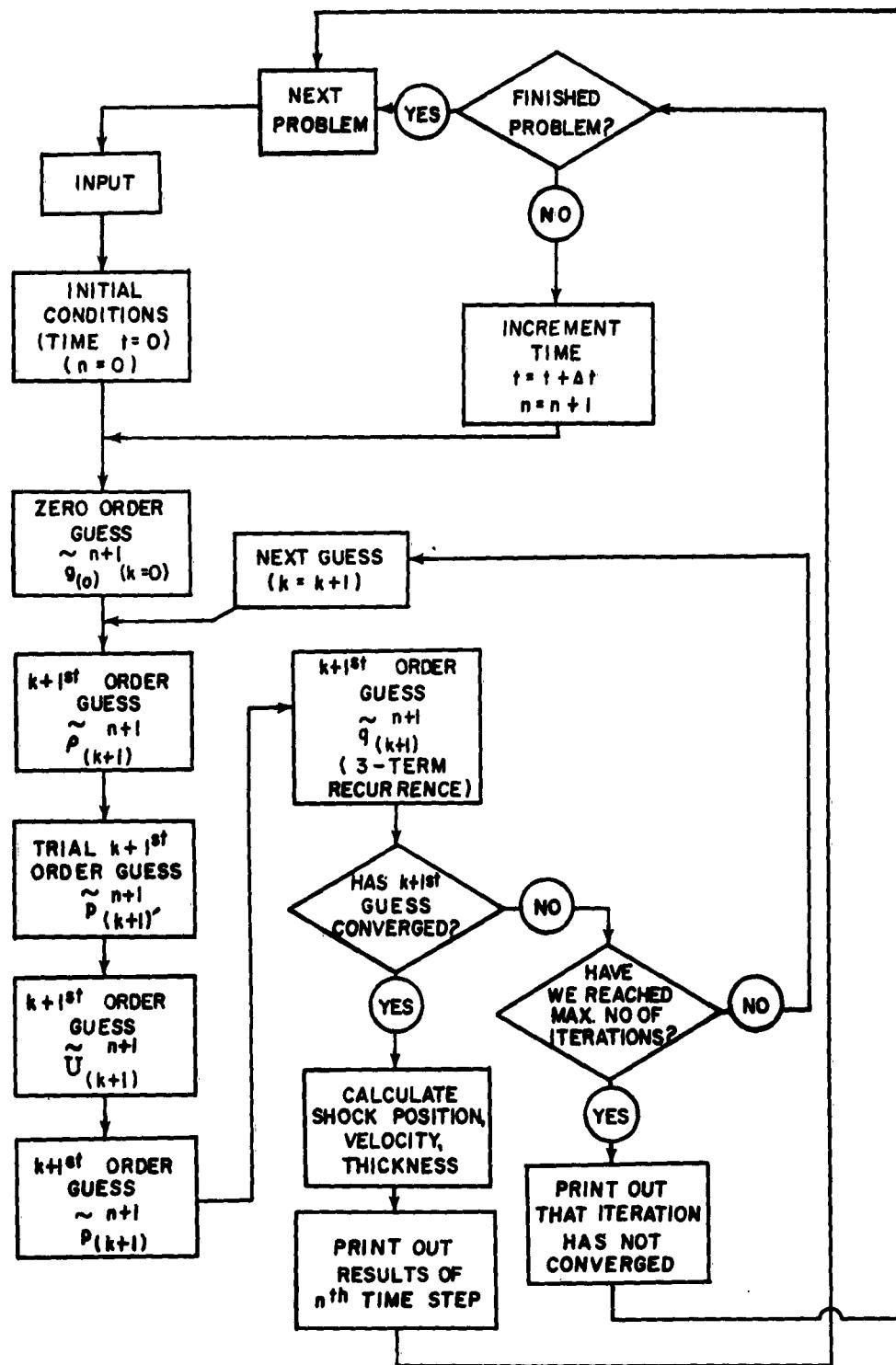


FIG. 8 FLOW CHART DEPICTING THE SEQUENCE OF CALCULATIONS IN THE IMPLICIT DIFFERENCE SCHEME

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General Electric Company, Space Sciences Laboratory, King of Prussia, Pennsylvania NUMERICAL INVESTIGATION OF ONE-DIMENSIONAL VISCO-PLASTIC MODEL Third Quarterly Report by T.D. Riney, October 1961. 34 p incl illus and tables (MSVD Reg. No. 214-796) (AFSC Project 9860) (Contract No. AF 08(635)-1713) (APGC-TN-61-43) Unclassified report

An explicit finite difference formulation of the equations governing the one-dimensional visco-plastic model was presented in the Second Quarterly Report. The scheme has been programmed on the IBM 7090 and exploratory calculations made for several values of the parameters μ_0 (viscosity factor), v_0 (yield stress) and v_0 (impact velocity). Excellent results are obtained for certain ranges of the parameters; these results are discussed in detail. For larger values of μ_0 and v_0 , however, the restriction on the size of Δt is very severe. A stability analysis is performed which explains this requirement. To circumvent this difficulty the governing equations are recast in an implicit finite difference scheme which is only valid for $\mu_0 > 0$, but which is then unconditionally stable. The method is described in detail and a flow chart is given for the machine program which is currently being written.

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